Structure formations and bifurcation points analysis in the accretion discs

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Abstract. It is presented basic points of the bifurcation, as one possible mechanism of instability. We have analyzed how this mechanism acts in accretion disc system, as we imply the behavior of reaction-diffusion systems and methods of their studying. We have shown that structures are formed after the arising of Rossby wave instability in the accretion disc.

Introduction

We may define the structure as a stable stationary state, that an irreversible transition from homogeneous equilibrium state to homogeneous stationary state is accomplished.

As a result of presentations [*Nicolis et al*, 1979; *Klimontovich*, 1996] we may suggest the statement: the spatial, temporal and spatial-temporal structures, which may arise far from equilibrium in non-linear area, when the parameters of the system exceed their critical values, calling – dissipative structures.

Pattern formation out of equilibrium can often be attributed to the coupling of diffusion with local nonlinear dynamic. Usually, these processes are explicit in a reaction-diffusion system, as it is shown in [*Borckmans et al*, *Andreeva et al*, 2002], where we have studied the structure formation and Turing instability. The arising of structures has a directly relation to the bifurcation solutions. In a definition, this is any qualitative or topology reconstruction of the system, when the parameter of the system crossed its critical value.

Here we concern our review over the action of the Turing and Hopf bifurcations, as a key mechanism to switch on instability or stability state.

The similar analysis are shown in the area of chemical reaction [*Nicolis*, 1994; *Engelhardt*, 1994], medicine and biology.

In this paper we present how these models work in astrophysics, especially in the accretion discs system.

Equations for bifurcation solutions in the accretion disc.

Now we will try to determine the kind of bifurcation mechanism, working in a non-axisymmetric, thin accretion disc.

We have presented in [*Andreeva et al*, 2002] the view of reaction - diffusion equations for Turing instability and we confirmed the necessary condition for its arising. We remind at first the view of these equations:

$$\frac{\partial \Psi_\varphi}{\partial t} = g(r,\varphi) + D_\varphi \nabla^2 \Psi_\varphi \qquad (1a)$$

$$\frac{\partial \Psi_r}{\partial t} = h(r,\varphi) + D_r \nabla^2 \Psi_r \qquad (1b)$$

Where Ψ is the vorticity in two directions; D is the diffusion coefficient; g and h are the source functions (or the reaction $\approx (\nabla.\Psi) v$);

Using the model of the chemical reactions, previous presents and eqs (1a, 1b), we may write the following equations:

$$\frac{\partial \Psi_r}{\partial t} = \Psi_r - \Psi_r^{\,3} - \Psi_\varphi + \frac{\partial^2 \Psi_r}{\partial r^2} \qquad (2a)$$

$$\frac{\partial \Psi_\varphi}{\partial t} = \frac{T_r}{T_\varphi}\left(\Psi_r - h(r,\varphi)\Psi_\varphi - g(r,\varphi)\right) + \frac{D_\varphi}{D_r}\frac{\partial^2 \Psi_\varphi}{\partial r^2} \qquad (2b)$$

For a more simple view and easy calculations of the above equations, let us make the next substitutions:

$$\frac{T_r}{T_\varphi} = \chi\,;\ \frac{D_\varphi}{D_r} = \delta\,;\ \Psi_r = j;\ \Psi_\varphi = q \qquad (3)$$

$$h(r,\varphi) = l_1$$

$$g(r,\varphi) = l_2$$

Then for the equations (2a,2b) comes after the following:

$$j_t = j - j^3 - q + j_{rr} \qquad (4a)$$

$$q_t = \chi\left(j - l_1 q - l_2\right) + \delta\, q_{rr} \qquad (4b)$$

The states, which are received from equilibrium by continuous disturbance, Prigozin [*Glensdorf et al*, 1973] calls thermodynamical branching.

If we ignore the derivatives on r and make some transformations of eqs. (4a,4b) then we have the following:

$q = j - j^3$ and $q = \left(j - l_2\right)/\,l_1$

The points of intersection of these two relations determine stationary homogeneous states, such at the dissipative structures.

Thus, there are three cases of intersection corresponding to different type of reaction-diffusion systems.

1) Intersecting at a single point, lying on one of the branch. / fig.1a/
2) Intersecting at a single point, lying on the middle branch. /fig 1b/
3) Intersecting at three points, each lying on a different branch. /fig 1c/

On the next figures, it is shown the graphicallv view of these relations :

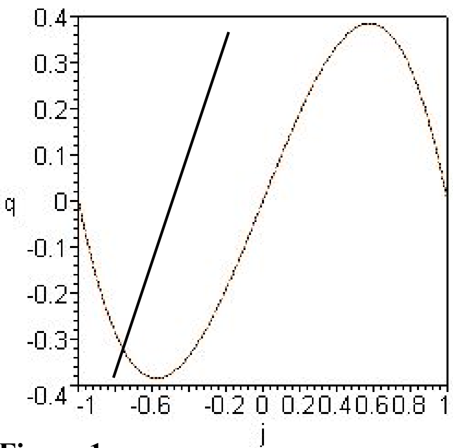

Figure 1a

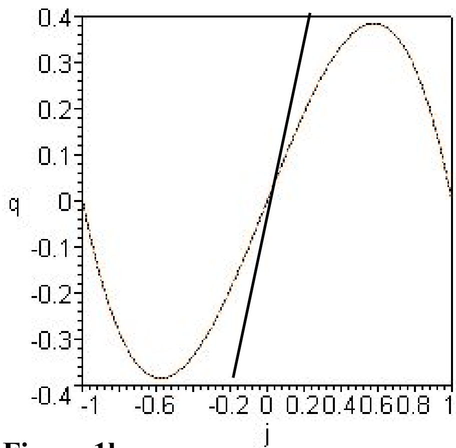

Figure 1b

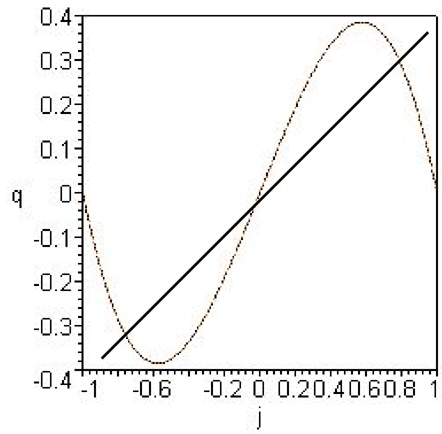

Figure 1c

In our view, we are interesting in a case 2, which is driving under Hopf and Turing bifurcations. The intersection points are independent of χ and δ , but these parameters, as we will see, affect on the stability, as a bifurcation parameters.

If we are following the symmetric FHN (FitzHugh Nagumo) [*Hagberg et al*, 1993] model, than $l_2 = 0$ and $0 < l_1 < 1$.

Thus the stationary homogeneous state, where $(j,q) = (0,0)$, loses stability in a Turing bifurcation when:

$$\lambda^{-1} = \lambda_{tk}^{-1} = 2 - l_1 + 2(1 - l_1)^{1/2} \qquad \left(\lambda = \chi/\delta\right) \qquad (5)$$
$$\chi > 1/l_1$$

This state, also, loses stability in a Hopf bifurcation when:

$$\chi = \chi_H = 1/l_1 \qquad (6)$$

But when the disc is a non-symmetric, we cannot neglect $l_2, l_2 \neq 0$, then the pitchfork bifurcation comes into saddle-node bifurcation. The pitchfork bifurcation occurs only when the parameters satisfy at least one equality. After the changes of parameter, this bifurcation disappears and we describe it as structurally unstable.

Thus, the couple action of two bifurcations, Hopf and Turing, leads to conditions for arising of spatio-temporal pattern formations.

Implication of bifurcation solutions in the Rossby wave instability.

When the parameter of the system reached its critical value, where the bifurcation acts, this system has two choices - to remain stable or proceed to instability. When the given system passed trough the bifurcation and lost its stability, there are conditions to forming structures in these places. What kind of structures appears in the accretion discs flow? The more often arising formations are spiral structures, vorticities, and solitons.

In the operation of Rossby wave instability the system has to undergo the necessary conditions. Since this instability is related to the entropy behavior, it is yield, so call, key function $\Re(r) = \Lambda(r) S^{2/\Gamma}(r)$, which has a maximum or a minimum. Than the instability is possible only if $\ln\left(\Lambda S^{2/\Gamma}\right)$ vanishes at some r.

Where $\Lambda \approx \frac{\Sigma\Omega}{k^2} = \frac{\Sigma}{2\Psi}$, S is the entropy of the system, Σ is the surface density, k is the wave number and $\Omega = v_\varphi / r$

The dispersion relation, which is valid here we may write in the form [*Lovelace et al*, 1999; *Morozov et al*, 2001]:

$$\Delta\omega = -\frac{k_\varphi\, c_s/\Omega}{1 + k^2 h^2}\left[(\ln \Re)' \pm \sqrt{[(\ln \Re)']^2 - \frac{1 + k^2 h^2}{L_s L_p}}\right] \qquad (7)$$

where L_s is the length scale of the entropy variation, L_p is the length scale of the prssure variation /we won't determine them here/, $k^2 = k_r^{\,2} + k_\varphi^{\,2}, h = c_s/\Omega$

From eq.(7) the maximum occures for $(\ln \Re)' = 0$ and it is:

$$\frac{\max(\omega_i)}{\Omega} = -\frac{|k_\varphi|\, c_s^{\,2}/\Omega^2}{\left(1 + k^2 h^2\right)^{1/2} \left(L_s L_p\right)^{1/2}} \approx \frac{|k_\varphi| h}{\left(1 + k^2 h^2\right)^{1/2}} \left(\frac{c_s}{v_k}\right)\left(\frac{r^2}{L_s L_p}\right)^{1/2} \qquad (8)$$

For a thin disc, the maximum rate is $< \Omega$.

Then, this maximum rate gives a possibility to arising a Rossby instability in the accretion disc.

Thus, after these wave braking and instabilities the Rossby vortices are formed.

On the next figures we show their development in the accretion discs and the view of Rossby waves, obtained independent of the equations above.

We perturbed the disk with a finite-amplitude perturbation. In the inner and outer adges there are free boundary conditions. We perturbed the disk velocity with a high order mode. The perturbation have a maximum amplitude of about 0,1-0,2 of the sound speed.

In fig.2 the initial rotation low is Keplerian. The perturbation induces waves in the disk and have time to propagate and form spiral waves in the disk flow. In this model the viscosity parameter $\alpha = 0,01$; the Mach nubmer $M = (H/r)^{-1} = 0,05$; After ≈5 Keprlerian orbits the waves smoothed out.

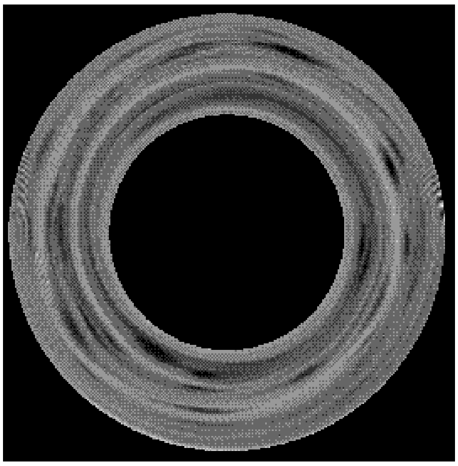

Figure 2. The waves and Rossby vortices in the accretion disc /dark spots/.

For the fig.3 we used $M = 0,15$ and $a = 5\times 10^{-5}$. Here the dissipation time of the perturbations are longer and the spiral waves are more clearly trace out. In this figure the voricity are received as the basic mark of instability.

In such way, we shown visually the places in the accretion discs, where the vortices may be formed. We used a specifically method and PC program to receiving this results.

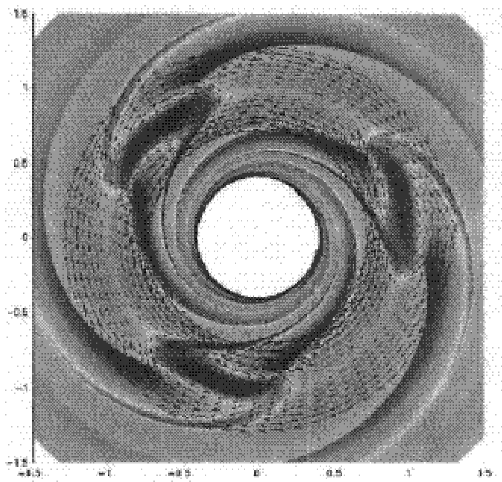

Figure 3. The vorticity formations on the spiral structure of the accretion disc.

Conclusion

The main effect of bifurcations is their relation to the appearing of structure formations in the accretion discs. We presented here two types of bifurcation solutions- the Hopf and the Turing -where the system loses its stability. We have shown that this instability causes the Rossby vortices, so the presence of such vortices would be crucial for the hydrodynamical transport of angular momentum in accretion discs. A wave of nonlinear Rossby vortices carries the mass and entropy maximum inward, exciting further vortices which transport the angular momentum outward. We know that the transfers of this kind is very important for the existence of the accretion discs.

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